

# Some notes about matrices, 4

Stephen William Semmes  
Rice University  
Houston, Texas

As usual, we let  $\mathbf{R}$  denote the real numbers,  $\mathbf{C}$  the complex numbers,  $\mathbf{R}^m$  the space of  $m$ -tuples of real numbers, and  $\mathbf{C}^n$  the space of  $n$ -tuples of real numbers. Also,  $\mathbf{Z}$  denotes the integers, and  $\mathbf{Z}^m$  the standard integer lattice in  $\mathbf{R}^m$ . We write  $\mathbf{Z}[i]$  for the *Gaussian integers*, which are the complex numbers of the form  $a + ib$  with  $a, b \in \mathbf{Z}$ , and  $(\mathbf{Z}[i])^n$  for the standard integer lattice in  $\mathbf{C}^n$ .

It will be convenient to write  $\mathcal{L}(\mathbf{R}^m, \mathbf{C}^n)$  for the space of real-linear mappings from  $\mathbf{R}^m$  to  $\mathbf{C}^n$ . The complex structure on  $\mathbf{C}^n$  is still relevant for this space, in that  $\mathcal{L}(\mathbf{R}^m, \mathbf{C}^n)$  is naturally a complex vector space. This is because one can multiply elements of  $\mathcal{L}(\mathbf{R}^m, \mathbf{C}^n)$  by  $i$ , and these linear transformations can be described by  $m \times n$  matrices of complex numbers in the usual manner, using the standard bases for  $\mathbf{R}^m$  and  $\mathbf{C}^n$ .

Let us write  $\mathcal{L}^*(\mathbf{R}^m, \mathbf{C}^n)$  for the subset of  $\mathcal{L}(\mathbf{R}^m, \mathbf{C}^n)$  consisting of linear transformations whose kernels are trivial, at least when  $m \leq 2n$ , so that this is possible. Using the usual Euclidean topology for  $\mathcal{L}(\mathbf{R}^m, \mathbf{C}^n)$ ,  $\mathcal{L}^*(\mathbf{R}^m, \mathbf{C}^n)$  is an open set. When  $m = 2n$ ,  $\mathcal{L}^*(\mathbf{R}^m, \mathbf{C}^n)$  consists of the invertible real-linear transformations from  $\mathbf{R}^m$  onto  $\mathbf{C}^n$ .

We can define a *lattice* in  $\mathbf{C}^n$  to be the image of  $\mathbf{Z}^{2n}$  under an element of  $\mathcal{L}^*(\mathbf{R}^{2n}, \mathbf{C}^n)$ . For such a lattice  $L$ , we get a quotient  $\mathbf{C}^n/L$  in the usual manner, which is both an abelian group under addition and a complex manifold, in fact a complex affine manifold. In other words, there are nice complex affine local coordinates for the quotient space coming from those of  $\mathbf{C}^n$ .

Of course the standard lattice  $(\mathbf{Z}[i])^n$  is a lattice in  $\mathbf{C}^n$  in this sense. For the moment let us restrict our attention to lattices  $L$  in  $\mathbf{C}^n$  which are of the form  $A((\mathbf{Z}[i])^n)$  for some invertible complex-linear mapping  $A$  on  $\mathbf{C}^n$ . This is a special case, in the same way that invertible complex-linear mappings on  $\mathbf{C}^n$  are a special case of elements of  $\mathcal{L}^*(\mathbf{R}^{2n}, \mathbf{C}^n)$ .

Let us write  $GL(\mathbf{C}^n)$  for the group of invertible complex-linear transformations on  $\mathbf{C}^n$ , and  $SL(\mathbf{C}^n)$  for the subgroup of  $GL(\mathbf{C}^n)$  consisting of linear transformations with determinant equal to 1. Also, we write  $U(\mathbf{C}^n)$  for the group of *unitary* linear transformations on  $\mathbf{C}^n$ , which are the invertible linear transformations which preserve the standard Hermitian inner product on  $\mathbf{C}^n$ , which is the same as saying that the inverse of the linear transformation is equal to its adjoint. The subgroup  $SU(\mathbf{C}^n)$  of  $U(\mathbf{C}^n)$  consists of the unitary linear transformations on  $\mathbf{C}^n$  which also have determinant equal to 1.

For the moment we are considering lattices in  $\mathbf{C}^n$  of the form  $A((\mathbf{Z}[i])^n)$ ,  $A \in GL(\mathbf{C}^n)$ . It is natural to look at these lattices up to unitary equivalence, which is to say that two lattices  $L_1, L_2$  are equivalent if there is a unitary linear transformation  $T$  on  $\mathbf{C}^n$  such that  $T(L_1) = L_2$ . This leads to an equivalence relation on  $GL(\mathbf{C}^n)$ , in which two invertible linear transformations  $A_1, A_2$  on  $\mathbf{C}^n$  are considered to be equivalent if there is a unitary linear transformation  $T$  on  $\mathbf{C}^n$  such that  $A_2 = T A_1$ .

The quotient of  $GL(\mathbf{C}^n)$  by this equivalence relation is denoted

$$(1) \quad U(\mathbf{C}^n) \backslash GL(\mathbf{C}^n).$$

This quotient space can be identified with the space of self-adjoint linear transformations on  $\mathbf{C}^n$  which are positive definite, through the mapping

$$(2) \quad A \in GL(\mathbf{C}^n) \mapsto A^* A.$$

That is, for each element  $A$  of  $GL(\mathbf{C}^n)$ , the product  $A^* A$  is a self-adjoint linear transformation on  $\mathbf{C}^n$  which is positive-definite,

$$(3) \quad A_1^* A_1 = A_2^* A_2$$

for two elements  $A_1, A_2$  of  $GL(\mathbf{C}^n)$  if and only if  $A_2 = T A_1$  for some unitary linear transformation  $T$  on  $\mathbf{C}^n$ , and every self-adjoint linear transformation on  $\mathbf{C}^n$  can be expressed as  $A^* A$  for some invertible linear transformation  $A$  on  $\mathbf{C}^n$ , and in fact has a unique self-adjoint positive-definite square root.

Similarly, one can consider two elements  $B_1, B_2$  of  $SL(\mathbf{C}^n)$  to be equivalent when there is a linear transformation  $U$  in the special unitary group  $SU(\mathbf{C}^n)$  such that  $B_2 = U B_1$ . The quotient  $SU(\mathbf{C}^n) \backslash SL(\mathbf{C}^n)$  can be identified with the space of self-adjoint linear transformations on  $\mathbf{C}^n$  which are positive-definite and have determinant 1, through the same mapping as before. Also, it will be convenient to restrict our attention for the moment

to lattices  $L$  of the form  $B((\mathbf{Z}[i])^n)$  for some  $B \in SL(\mathbf{C}^n)$ , which is just a modest additional normalization.

Let us write  $\Sigma(\mathbf{C}^n)$  for the subgroup of  $SL(\mathbf{C}^n)$  of linear transformations whose associated  $n \times n$  matrices, with respect to the standard basis for  $\mathbf{C}^n$ , have integer entries, which implies that the matrices associated to their inverses also have integer entries. Thus  $B((\mathbf{Z}[i])^n) = (\mathbf{Z}[i])^n$  when  $B \in \Sigma(\mathbf{R}^n)$ , and conversely  $B \in SL(\mathbf{C}^n)$  and  $B((\mathbf{Z}[i])^n) = (\mathbf{Z}[i])^n$  implies that  $B \in \Sigma(\mathbf{C}^n)$ . The quotient  $SL(\mathbf{C}^n)/\Sigma(\mathbf{C}^n)$  represents the space of lattices that we are considering at the moment, the double quotient  $SU(\mathbf{C}^n) \backslash SL(\mathbf{C}^n)/\Sigma(\mathbf{C}^n)$  represents the space of these lattices modulo equivalence under special unitary transformations, and this double quotient can also be identified with the quotient of the space of self-adjoint positive-definite linear transformations on  $\mathbf{C}^n$  with determinant 1 by the action of  $\Sigma(\mathbf{C}^n)$  defined by  $P \mapsto B^* P B$ ,  $B \in \Sigma(\mathbf{C}^n)$ .

Now let us look at general lattices in  $\mathbf{C}^n$ , under the equivalence relation in which two lattices  $L_1, L_2$  are considered to be equivalent if there is an invertible complex-linear transformation  $A$  on  $\mathbf{C}^n$  such that  $A(L_1) = L_2$ . This leads to an equivalence relation on  $\mathcal{L}^*(\mathbf{R}^{2n}, \mathbf{C}^n)$ , in which two elements of  $\mathcal{L}^*(\mathbf{R}^{2n}, \mathbf{C}^n)$  are considered to be equivalent if one can be written as the composition of an invertible complex-linear transformation on  $\mathbf{C}^n$  with the other element of  $\mathcal{L}^*(\mathbf{R}^{2n}, \mathbf{C}^n)$ . In other words, we look at the action of  $GL(\mathbf{C}^n)$  on  $\mathcal{L}^*(\mathbf{R}^{2n}, \mathbf{C}^n)$  by post-composition.

Actually, it is more convenient to consider  $\mathcal{L}_1^*(\mathbf{R}^{2n}, \mathbf{C}^n)$ , which we define to be the subset of  $\mathcal{L}^*(\mathbf{R}^{2n}, \mathbf{C}^n)$  consisting of invertible real-linear transformations from  $\mathbf{R}^{2n}$  to  $\mathbf{C}^n$  such that the image of the first  $n$  standard basis vectors in  $\mathbf{R}^{2n}$  are linearly-independent over the complex numbers as  $n$  vectors in  $\mathbf{C}^n$ . This restriction is not too serious, and indeed we can describe the lattices in  $\mathbf{C}^n$  as images of  $\mathbf{Z}^{2n}$  under mappings in  $\mathcal{L}_1^*(\mathbf{R}^{2n}, \mathbf{C}^n)$ . In other words, if we start with a lattice  $L$  given as the image of  $\mathbf{Z}^{2n}$  under an element of  $\mathcal{L}^*(\mathbf{R}^{2n}, \mathbf{C}^n)$ , we can rewrite it as the image of  $\mathbf{Z}^{2n}$  under a linear transformation in  $\mathcal{L}_1^*(\mathbf{R}^{2n}, \mathbf{C}^n)$  by pre-composing the initial linear transformation from  $\mathbf{R}^{2n}$  to  $\mathbf{C}^n$  with an invertible linear transformation on  $\mathbf{R}^{2n}$  which permutes the standard basis vectors in a suitable way.

To deal with the action of  $GL(\mathbf{C}^n)$  by post-composition, we can restrict ourselves to  $\mathcal{L}^{**}(\mathbf{R}^{2n}, \mathbf{C}^n)$ , which we define to be the space of invertible real-linear transformations from  $\mathbf{R}^{2n}$  to  $\mathbf{C}^n$  such that the images of the first  $n$  standard basis vectors in  $\mathbf{R}^{2n}$  are the  $n$  standard basis vectors in  $\mathbf{C}^n$ , and in the same order. In other words, if we identify  $\mathbf{R}^{2n}$  with the Cartesian

product  $\mathbf{R}^n \times \mathbf{R}^n$ , then these are the invertible real-linear transformations from  $\mathbf{R}^n \times \mathbf{R}^n$  onto  $\mathbf{C}^n$  with the property that on  $\mathbf{R}^n \times \{0\}$  they coincide with the standard embedding of  $\mathbf{R}^n$  into  $\mathbf{C}^n$ . This exactly compensates for the action of  $GL(\mathbf{C}^n)$  by post-composition, since for any collection  $v_1, \dots, v_n$  of linearly-independent vectors in  $\mathbf{C}^n$  there is a unique  $A \in GL(\mathbf{C}^n)$  such that  $A(v_1), \dots, A(v_n)$  are the standard basis vectors in  $\mathbf{C}^n$ , in order.

We can identify  $\mathcal{L}^{**}(\mathbf{R}^{2n}, \mathbf{C}^n)$  with an open subset of  $\mathcal{L}(\mathbf{R}^n, \mathbf{C}^n)$ . That is, elements of  $\mathcal{L}^{**}(\mathbf{R}^{2n}, \mathbf{C}^n)$  can be identified with linear transformations from  $\mathbf{R}^n \times \mathbf{R}^n$  into  $\mathbf{C}^n$ , and these linear transformations are determined by what they do on  $\{0\} \times \mathbf{R}^n$ , since their behavior on  $\mathbf{R}^n \times \{0\}$  is fixed by definition. We can think of elements of  $\mathcal{L}(\mathbf{R}^n, \mathbf{C}^n)$  as being written as  $A + iB$ , where  $A, B$  are linear transformations on  $\mathbf{R}^n$ , and one can check that the elements of  $\mathcal{L}^{**}(\mathbf{R}^{2n}, \mathbf{C}^n)$  correspond exactly to elements of  $\mathcal{L}(\mathbf{R}^n, \mathbf{C}^n)$  of the form  $A + iB$ , where  $A, B$  are linear transformations on  $\mathbf{R}^n$  and  $B$  is invertible.

To be more precise, it is helpful to think in terms of real-linear mappings on  $\mathbf{C}^n$ , which can be written as

$$(4) \quad T(x + iy) = E_1(x) + E_2(y) + i(E_3(x) + E_4(y)),$$

where  $x, y \in \mathbf{R}^n$ . The passage to  $\mathcal{L}_1^*(\mathbf{R}^{2n}, \mathbf{C}^n)$  can be expressed in these terms as the restriction to invertible real-linear transformations  $T$  on  $\mathbf{C}^n$  of the form

$$(5) \quad T(x + iy) = x + A(y) + iB(y),$$

where  $A, B$  are linear transformations on  $\mathbf{R}^n$ . The condition of invertibility of  $T$  is equivalent to the invertibility of  $B$  on  $\mathbf{C}^n$ .

Another way to look at real-linear mappings on  $\mathbf{C}^n$  is as mappings of the form

$$(6) \quad T(z) = M(z) + \overline{N(z)},$$

where  $z \in \mathbf{C}^n$ ,  $M$  and  $N$  are complex-linear mappings on  $\mathbf{C}^n$ , and for  $w \in \mathbf{C}^n$ ,  $\overline{w}$  is the element of  $\mathbf{C}^n$  whose coordinates are the complex-conjugates of the coordinates of  $w$ .

Invertibility of  $T$  is a bit tricky, and as an important special case, it is natural to restrict our attention to mappings  $T$  as above for which  $M$  majorizes  $N$  in the sense that

$$(7) \quad |N(z)| < |M(z)|$$

for  $z \in \mathbf{C}^n$ ,  $z \neq 0$ , where  $|w|$  denotes the standard Euclidean norm of  $w \in \mathbf{C}^n$ . To factor out the action of  $GL(\mathbf{C}^n)$  by post-composition, we can restrict our

attention to real-linear transformations  $T$  of the form

$$(8) \quad T(z) = z + \overline{E(z)},$$

where  $E$  is a complex-linear transformation on  $\mathbf{C}^n$  with operator norm strictly less than 1, which is equivalent to saying that  $E^* E < I$ . This has nice features when we think of the image of the standard integer lattice  $(\mathbf{Z}[i])^n$  under  $T$ , with points in the image being reasonably-close to their counterparts in the original lattice.

The  $n = 1$  case is quite instructive. We can write a real-linear transformation  $T$  on  $\mathbf{C}$  as

$$(9) \quad T(x + i y) = a x + i b y$$

for  $x, y \in \mathbf{R}$ , where  $a, b$  are complex numbers, and when  $T$  is invertible we can rewrite this as

$$(10) \quad T(x + i y) = a(x + i c y),$$

where  $a, c$  are complex numbers with  $a \neq 0$  and  $c$  having nonzero imaginary part. Alternatively, we can write a real-linear transformation  $T$  on  $\mathbf{C}$  as  $T(z) = \alpha z + \beta \bar{z}$  with  $\alpha, \beta \in \mathbf{C}$ , and where  $T$  is invertible if and only if  $|\alpha| \neq |\beta|$ , and when  $|\alpha| > |\beta|$  this can be rewritten as

$$(11) \quad T(z) = \theta(z + \mu \bar{z}),$$

where  $\theta$  is a nonzero complex number and  $\mu$  is a complex number such that  $|\mu| < 1$ .